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TRANSFER FUNCTIONS AND BLOCK DIAGRAMS

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2. Transfer Function of Linear Time-Invariant (LTI) Systems

3. Block Diagrams

4. Multiple Inputs

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1. Introduction

An important step in the analysis and design of control systems is the mathematical modelling of the controlled process. There are a number of mathematical representations to describe a controlled process:

Differential equations: You have learned this before.

Transfer function: It is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all zero initial conditions.

Block diagram: It is used to represent all types of systems. It can be used, together with transfer functions, to describe the cause and effect relationships throughout the system.

State-space-representation: You will study this in an advanced Control Systems Design course.

1.1. Linear Time-Variant and Linear Time-Invariant Systems

Definition 1: A *time-variable differential equation* is a differential equation with one or more of its coefficients are functions of time, t. For example, the differential equation:

$$t^2 \frac{d^2 y(t)}{dt^2} + y(t) = u(t)$$

(where u and y are dependent variables) is time-variable since the term $t^2 d^2 y/dt^2$ depends explicitly on t through the coefficient t^2.

An example of a *time-varying system* is a spacecraft system which the mass of spacecraft changes during flight due to fuel consumption.

Definition 2: A *time-invariant differential equation* is a differential equation in which none of its coefficients depend on the independent time variable, t. For example, the differential equation:

$$m \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + y(t) = u(t)$$

where the coefficients m and b are constants, is time-invariant since the equation depends only implicitly on t through the dependent variables y and u and their derivatives.

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Dynamic systems that are described by linear, constant-coefficient, differential equations are called *linear time-invariant* (LTI) *systems*.

2. Transfer Function of Linear Time-Invariant (LTI) Systems

The transfer function of a linear, time-invariant system is defined as the ratio of the Laplace transform of the output (response function), $Y(s) = \mathcal{L}\{y(t)\}$, to the Laplace transform of the input (driving function) $U(s) = \mathcal{L}\{u(t)\}$, under the assumption that all initial conditions are zero.

$$u(t) \longrightarrow \boxed{\text{System differential equation}} \longrightarrow y(t)$$
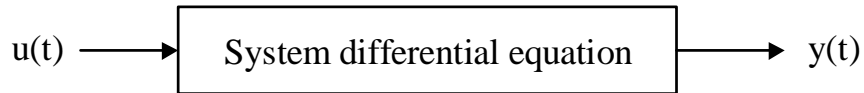
Taking the Laplace transform with zero initial conditions,

$$U(s) \longrightarrow \boxed{\text{System transfer function}} \longrightarrow Y(s)$$
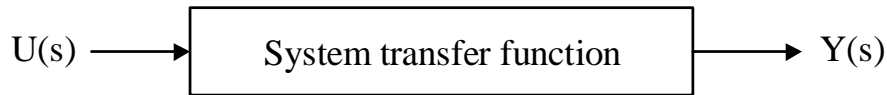
Transfer function: $G(s) = \dfrac{Y(s)}{U(s)}$

A dynamic system can be described by the following time-invariant differential equation:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

Taking the Laplace transform and considering zero initial conditions we have:

$$\left(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0\right) Y(s) = \left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0\right) U(s)$$

The transfer function between u(t) and y(t) is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{M(s)}{N(s)}$$

where $G(s) = M(s)/N(s)$ is the transfer function of the system; the roots of N(s) are called *poles* of the system and the roots of M(s) are called *zeros* of the system. By setting the denominator function to zero, we obtain what is referred to as the *characteristic equation*:

$$a_n s_n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

We shall see later that the stability of linear, SISO systems is completely governed by the roots of the characteristic equation.

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A transfer function has the following properties:
- The transfer function is defined only for a linear time-invariant system. It is not defined for nonlinear systems.
- The transfer function between a pair of input and output variables is the ratio of the Laplace transform of the output to the Laplace transform of the input.
- All initial conditions of the system are set to zero.
- The transfer function is independent of the input of the system.

To derive the transfer function of a system, we use the following procedures:
1. Develop the differential equation for the system by using the physical laws, e.g. Newton's laws and Kirchhoff's laws.
2. Take the Laplace transform of the differential equation under the zero initial conditions.
3. Take the ratio of the output Y(s) to the input U(s). This ratio is the transfer function.

Example: Consider the following RC circuit.
1) Find the transfer function of the network, $V_o(s)/V_i(s)$.
2) Find the response $v_o(t)$ for a unit-step input, i.e.

$$v_i(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Solution:
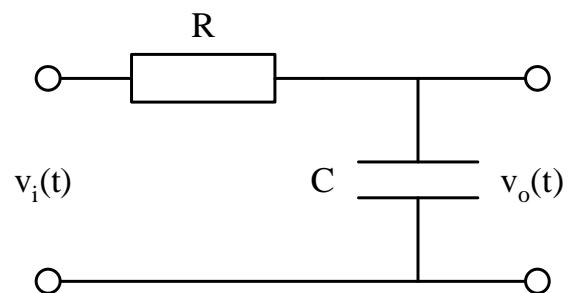
Exercise: Consider the LCR electrical network shown in the figure below. Find the transfer function $G(s) = V_o(s)/V_i(s)$.
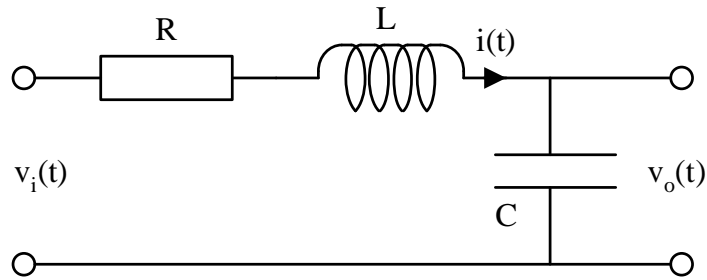
Exercise: Find the time response of $v_o(t)$ of the above system for $R = 2.5\Omega$, $C = 0.5F$, $L=0.5H$ and

$$v_i(t) = \begin{cases} 0 & t < 0 \\ 2 & t \geq 0 \end{cases}.$$

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Exercise: In the mechanical system shown in the figure, m is the mass, k is the spring constant, b is the friction constant, u(t) is an external applied force and y(t) is the resulting displacement.
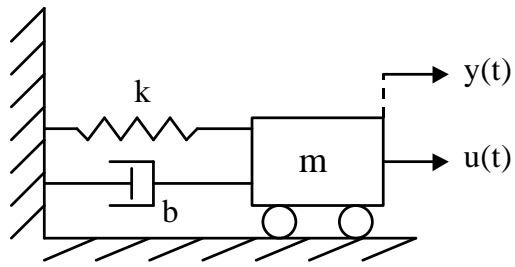
1) Find the differential equation of the system
2) Find the transfer function between the input U(s) and the output Y(s).

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3. Block Diagrams

A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals. The block diagram gives an overview of the system.

Block diagram items:
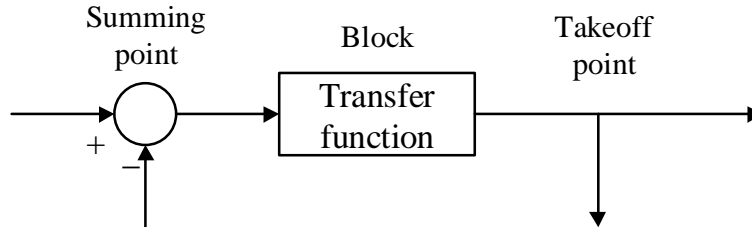
The above figure shows the way the various items in block diagrams are represented. Arrows are used to represent the directions of signal flow. A summing point is where signals are algebraically added together. The takeoff point is similar to the electrical circuit takeoff point. The block is usually drawn with its transfer funciton written inside it.

We will use the following terminology for block diagrams throughout this course:

R(s) = reference input (command)
Y(s) = output (controlled variable)
U(s) = input (actuating signal)
E(s) = error signal
F(s) = feedback signal
G(s) = forward path transfer function
H(s) = feedback transfer fucntion
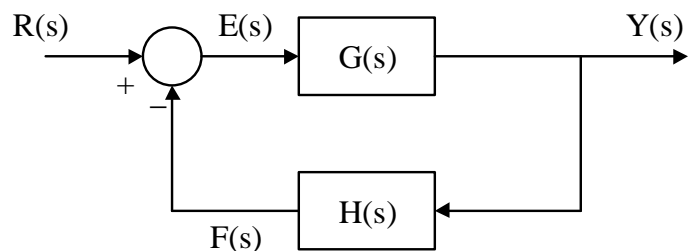
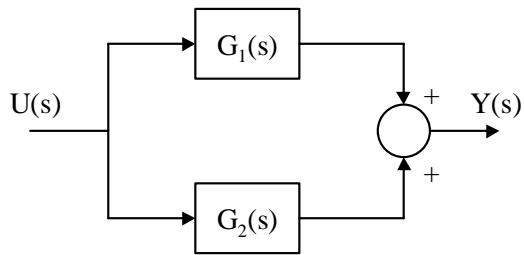

Single block:
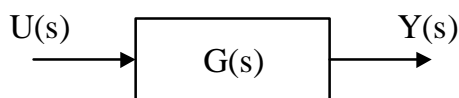
$$Y(s) = G(s)U(s)$$

U(s) is the input to the block, Y(s) is the output of the block and G(s) is the transfer function of the block.
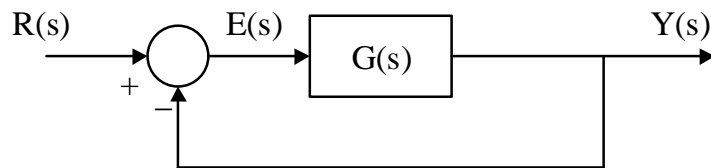
Series connection:
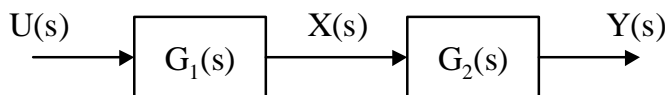
$$Y(s) = G_1(s)G_2(s)U(s)$$

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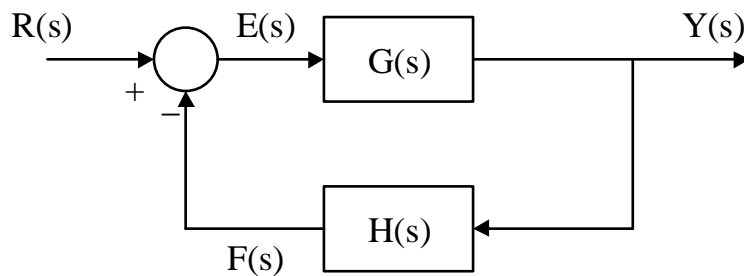
Parallel connection (feed forward):

$$Y(s) = [G_1(s) + G_2(s)]U(s)$$

Negative feedback system (closed-loop system):

The closed loop transfer function: $$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

**Exercise**: Find the closed-loop transfer function for the following block diagram:

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Exercise: A control system has a forward path of two elements with transfer functions K and $1/(s+1)$ as shown. If the feedback path has a transfer function s, what is the transfer function of the closed loop system.
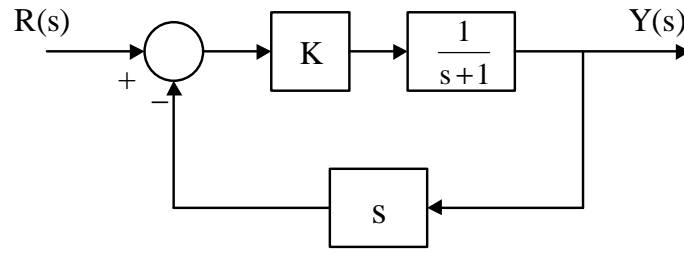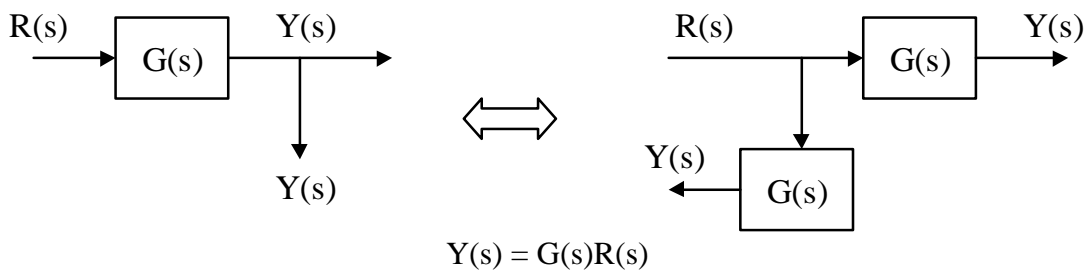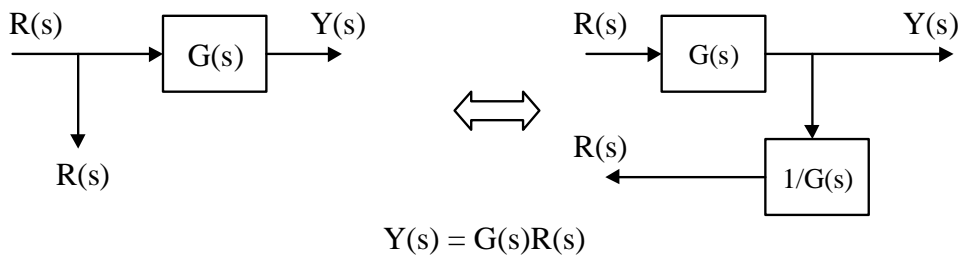

Moving a summing point ahead of a block:

$$Y(s) = G(s)R(s) \pm F(s)$$
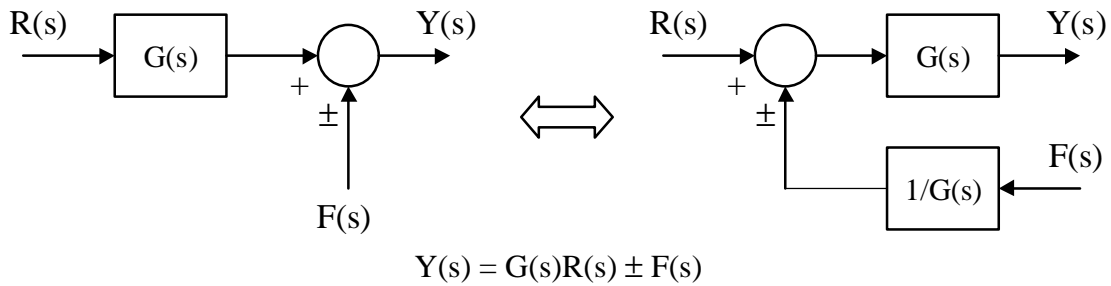
Moving a summing point beyond a block:

$$Y(s) = G(s)[R(s) \pm F(s)]$$
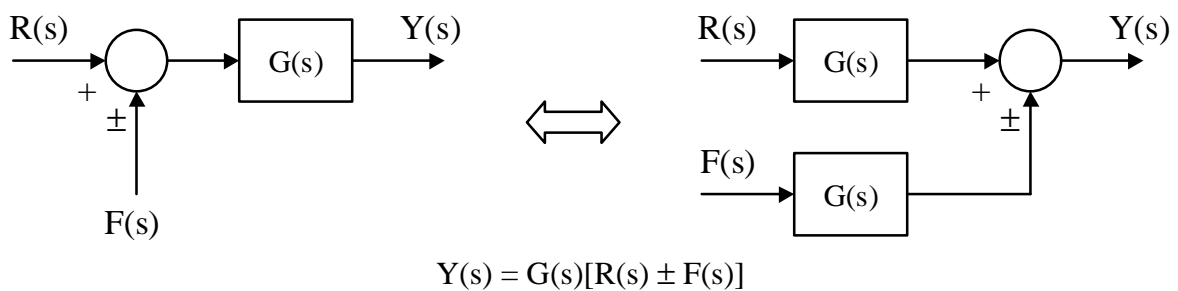
Moving a takeoff point ahead of a block:

$$Y(s) = G(s)R(s)$$

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Moving a takeoff point beyond a block:

$$Y(s) = G(s)R(s)$$

Moving a takeoff point ahead of a summing point:

$$Y(s) = R(s) \pm F(s)$$
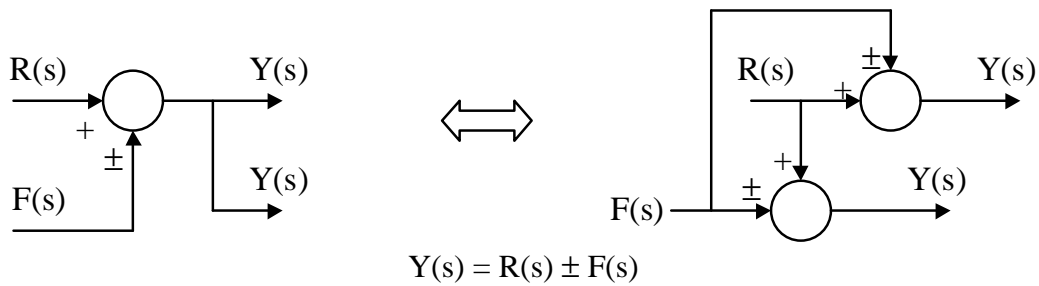
Moving a takeoff point beyond a summing point:

$$Y(s) = R(s) \pm F(s)$$
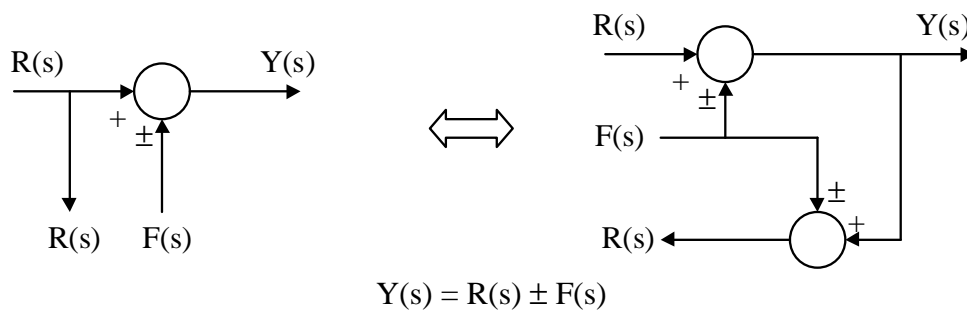
Exercise: Reduce the following block diagram and determine the transfer function.

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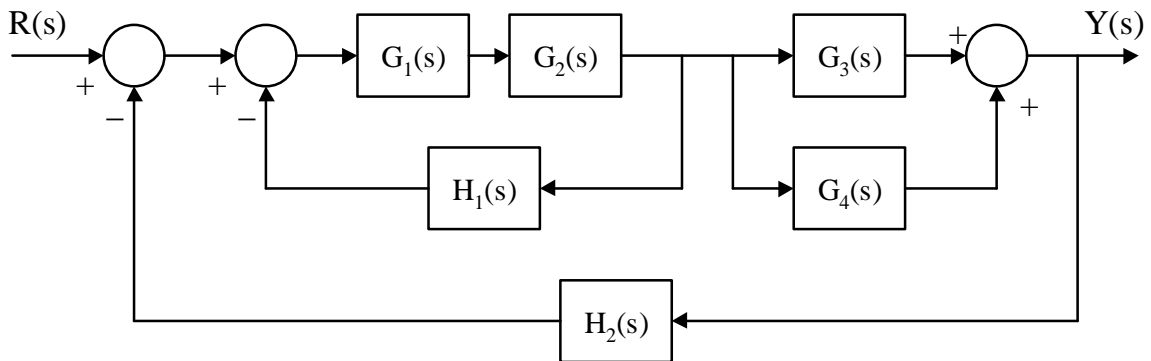
Exercise: Reduce the following block diagram and determine the transfer function.

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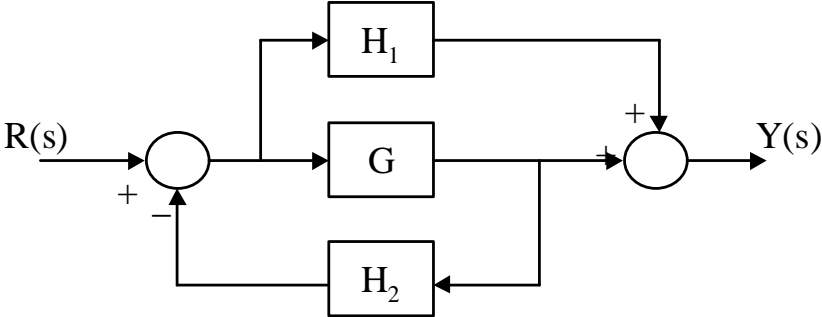
4. Multiple Inputs

Control systems often have more than one input. For example, there can be the input signal indicating the required value of the controlled variable and also an input or inputs due to disturbances which affect the system. The procedure to obtain the relationship between the inputs and the output for such systems is:

1. Set all inputs except one equal to zero
2. Determine the output signal due to this one non-zero input
3. Repeat the above steps for each of the remaining inputs in turn
4. The total output of the system is the algebraic sum (superposition) of the outputs due to each of the inputs.

Example: Find the output Y(s) of the block diagram in the figure below.
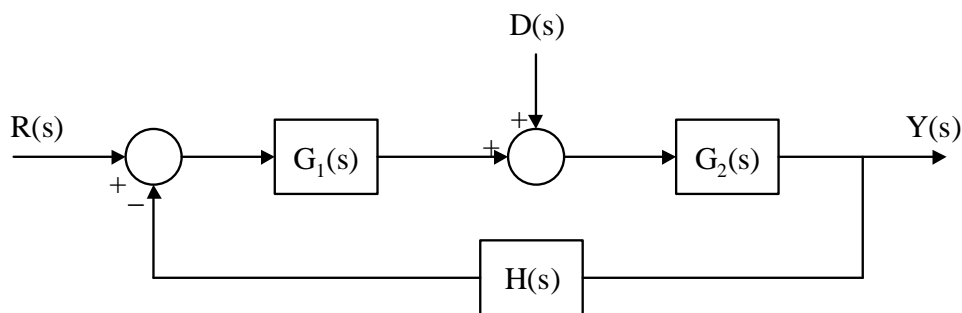
Solution:

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Exercise: Determine the output Y(s) of the following system.

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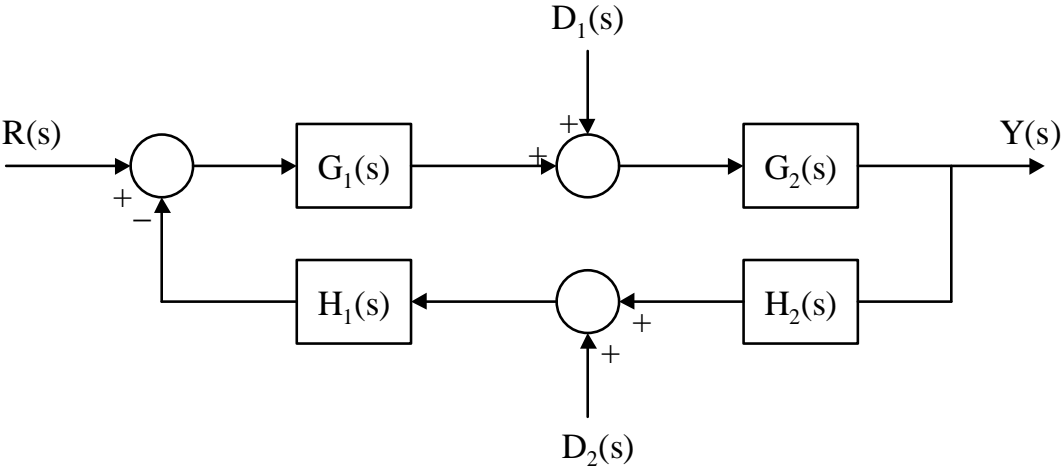
5. Transfer Functions with MATLAB

A transfer function of a linear time-invariant (LTI) system can be entered into MATLAB using the command **tf(num,den)** where **num** and **den** are row vectors containing, respectively, the coefficients of the numerator and denominator polynomials of the transfer function. For example, the transfer function:
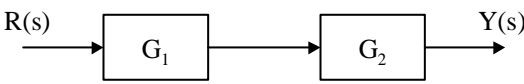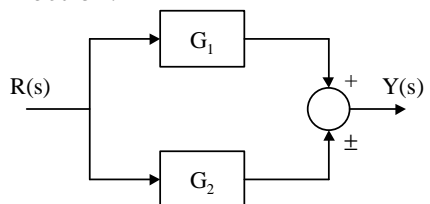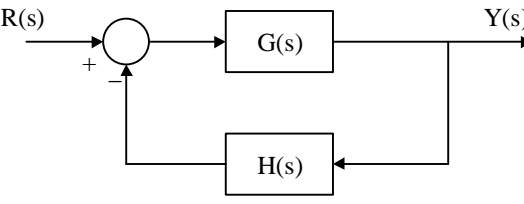
$$G(s) = \frac{3s+1}{s^2 + 3s + 2}$$

can be entered into MATLAB by typing the following on the command line:

```
num = [3 1];
den = [1 3 2];
G = tf(num,den)
```

The output on the MATLAB command window would be:

```
Transfer function:
   3 s + 1
-------------
s^2 + 3 s + 2
```

Once the various transfer functions have been entered, you can combine them together using arithmetic operations such as addition and multiplication to evaluate the transfer function of a cascaded system. The following table lists the most common systems connections and the corresponding MATLAB commands to implement them. In the following, **SYS** refers to the transfer function of a system, i.e. **SYS** = Y(s)/R(s).

System	MATLAB command
Series connection: R(s) → G_1 → G_2 → Y(s)	**SYS = G1*G2** or **SYS = series(G1,G2)**
Parallel connection: R(s) splits to G_1 and G_2, + Y(s) $\pm$	**SYS = G1 ± G2** or **SYS = parallel(G1,±G2)**
Negative feedback connection: R(s) → + − → G(s) → Y(s), feedback H(s)	**SYS = feedback(G,H)**

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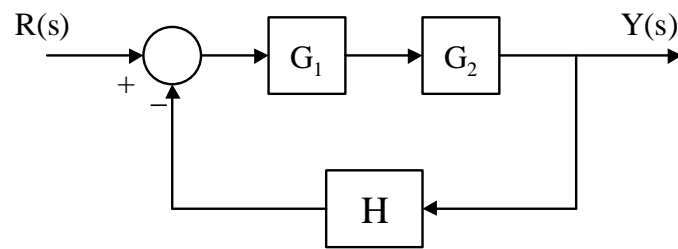
Example: Evaluate the transfer function of the feedback system shown in the figure above using MATLAB where $G_1(s) = 4$, $G_2(s) = 1/(s+2)$ and $H(s) = 5s$.

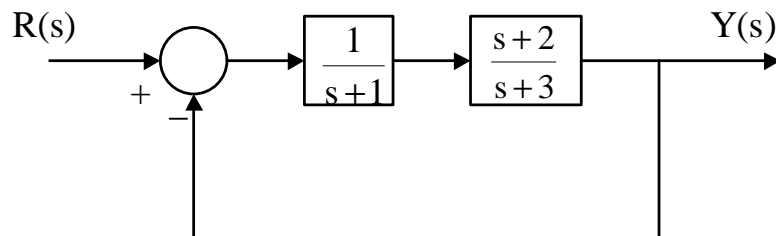Solution: Type the following in the MATLAB command line:

```
G1 = tf([0 4],[0 1]);
G2 = tf([0 1],[1 2]);
H = tf([5 0],[0 1]);
SYS = feedback(G1*G2,H)
```

This produces the following output on the command window (check this result):

```
Transfer function:
     4
 --------
 21 s + 2
```

Exercise: Compute the closed-loop transfer function of the following system using MATLAB.

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6. Time Response Analysis with MATLAB

After entering the transfer function of a LTI system, we can compute and plot the time response of this system due to different input stimuli in MATLAB. In particular, we will consider the step response, the impulse response, the ramp response, and responses to other simple inputs.

6.1. Step response

To plot the unit-step response of the LTI system **SYS=tf(num,den)** in MATLAB, we use the command **step(SYS)**. We can also enter the row vectors of the numerator and denominator coefficients of the transfer function directly into the **step** function: **step(num,den)**.

Example: Plot the unit-step response of the following system in MATLAB:

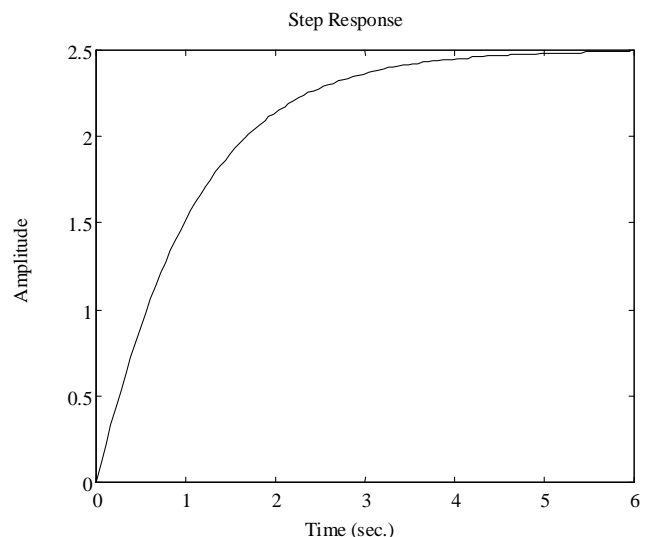$$\frac{Y(s)}{R(s)} = \frac{2s+10}{s^2+5s+4}$$

Solution:

```
num = [0 2 10];
den = [1 5 4];
SYS = tf(num,den);
step(SYS)
```

or directly:

```
step(num,den)
```

MATLAB will then produce the following plot on the screen. Confirm this plot yourself.

For a step input of magnitude other than unity, for example K, simply multiply the transfer function **SYS** by the constant **K** by typing **step(K*SYS)**. For example, to plot the response due to a step input of magnitude 5, we type **step(5*SYS)**.

Notice in the previous example that that time axis was scaled automatically by MATLAB. You can specify a different time range for evaluating the output response. This is done by first defining the required time range by typing:

```
t = 0:0.1:10;    % Time axis from 0 sec to 10 sec in steps of 0.1 sec
```

and then introducing this time range in the **step** function as follows:

```
step(SYS,t)      % Plot the step response for the given time range, t
```

This produces the following plot for the same example above.

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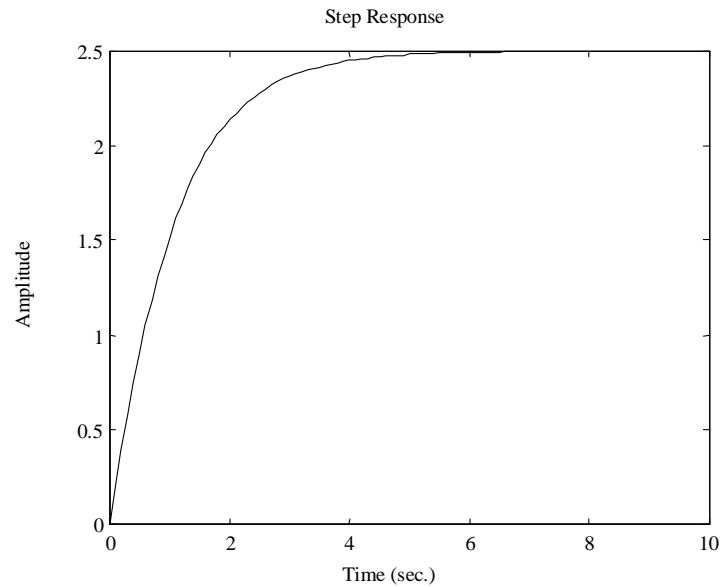
Step Response

You can also use the **step** function to plot the step responses of multiple LTI systems **SYS1**, **SYS2**, ... etc. on a single figure in MATLAB by typing:

```
step(SYS1,SYS2,...)
```

6.2. Impulse response

The unit-impulse response of a control system **SYS=tf(num,den)** may be plotted in MATLAB using the function **impulse(SYS)**.

Example: Plot the unit-impulse response of the following system in MATLAB:
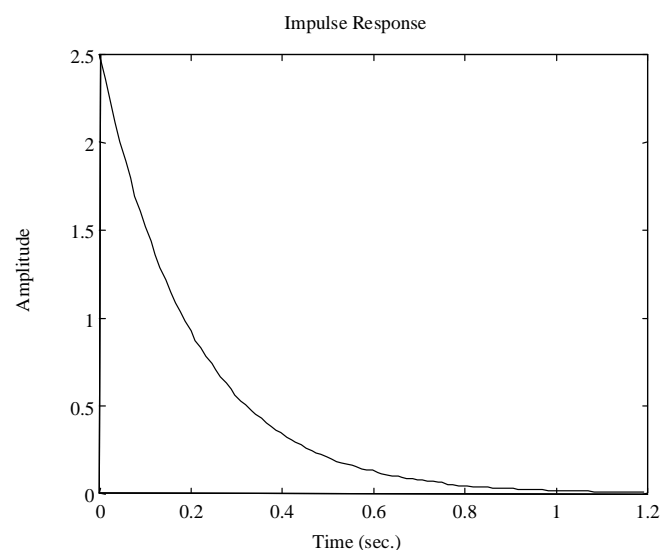
$$\frac{Y(s)}{R(s)} = \frac{5}{2s+10}$$

Solution:

```
num = [0 5];
den = [2 10];
SYS = tf(num,den);
impulse(SYS)
```

or directly

```
impulse(num,den)
```

This will produce the following output on the screen. Is that what you would expect?

Impulse Response

6.3. Ramp response

There is no ramp command in MATLAB. To obtain the unit ramp response of the transfer function G(s):

- multiply G(s) by 1/s, and
- use the resulting function in the **step** command.

The step command will further multiply the transfer function by 1/s to make the input $1/s^2$ i.e. Laplace transform of a unit-ramp input. For example, consider the system:

$$\frac{Y(s)}{R(s)} = \frac{1}{s^2 + s + 1}$$

With a unit-ramp input, $R(s) = 1/s^2$, the output can be written in the form:

$$Y(s) = \frac{1}{s^2 + s + 1} R(s) = \frac{1}{(s^2 + s + 1)s} \cdot \frac{1}{s}$$

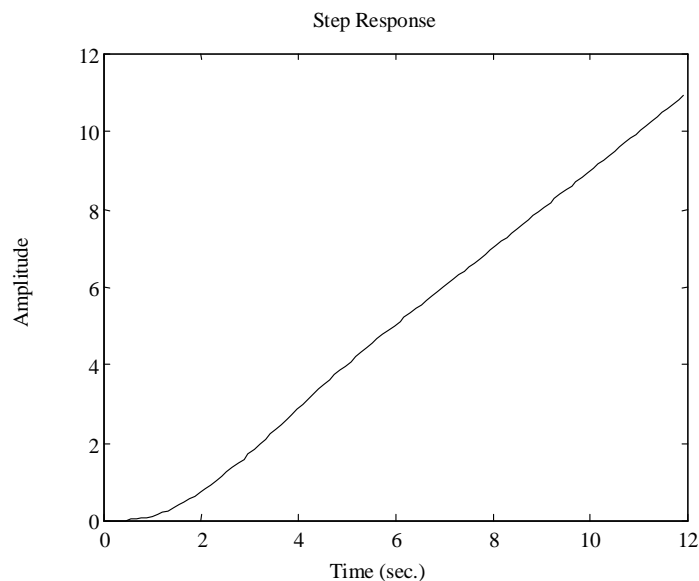$$= \left[\frac{1}{s^3 + s^2 + s} \right] \cdot \frac{1}{s}$$

which is equivalent to multiplying by 1/s and then working out the step response. To plot the unit-ramp response of this system, we enter the numerator and denominator coefficients of the term in square brackets into MATLAB:

```
num = [0 0 0 1];
den = [1 1 1 0];
```

and use the **step** command:

```
step(num,den)
```

The unit ramp response will be plotted by MATLAB as shown below.

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6.4. Arbitrary response

To obtain the time response of the LTI system **SYS=tf(num,den)** to an arbitrary input (e.g. exponential function, sinusoidal function .. etc.), we can use the **lsim** command (stands for 'linear simulation') as follows:

lsim(SYS,r,t) or **lsim(num,den,r,t)**

where **num** and **den** are the row vectors of the numerator and denominator coefficients of the transfer function, **r** is the input time function, and **t** is the time range over which **r** is defined.

Example: Use MATLAB to obtain the output time response of the transfer function:

$$\frac{Y(s)}{R(s)} = \frac{2}{s+3}$$

when the input r is given by $r = e^{-t}$.

Solution: Start by entering the row vectors of the numerator and denominator coefficients in MATLAB:
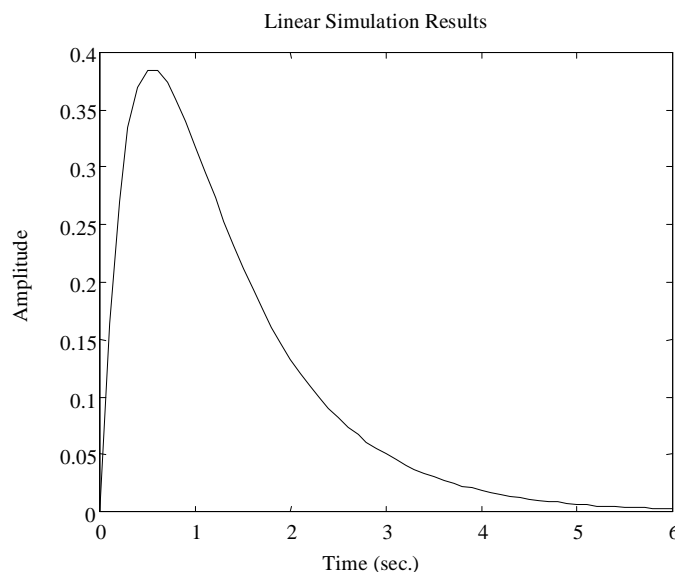
```
num = [0 2];
den = [1 3];
```

Then specify the required time range and define the input function, r, over this time:

```
t = 0:0.1:6;     % Time range from 0 to 6 sec in steps of 0.1 sec
r = exp(-t);     % Input time function
```

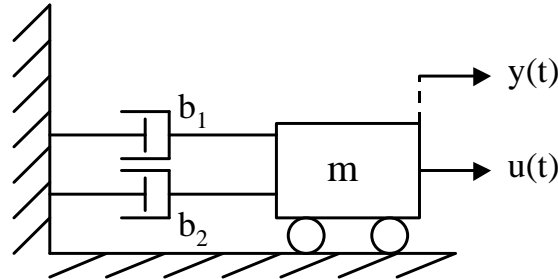Enter the above information into the **lsim** function by typing:

```
lsim(num,den,r,t)
```

This would produce the following plot on the screen.

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TUTORIAL PROBLEM SHEET 3

1. Find the transfer function between the input force u(t) and the output displacement y(t) for the system shown below.


where b_1 and b_2 are the frictional coefficients. For b_1 = 0.5 N-s/m, b_2 = 1.5 N-s/m, m = 10 kg and u(t) is a unit-impulse function, what is the response y(t)? Check and plot the response with MATLAB.

2. For the following circuit, find the transfer function between the output voltage across the inductor y(t), and the input voltage u(t).
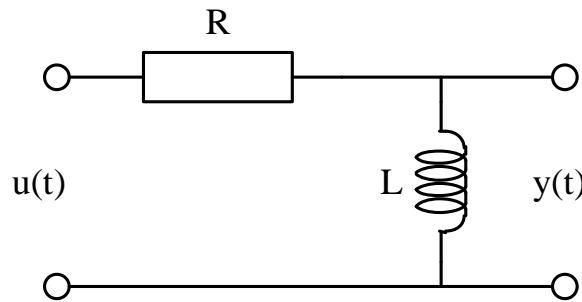
For R = 1 Ω, L = 0.1 H, and u(t) is a unit-step function, what is the response y(t)? Check and plot the result using MATLAB.

3. Find the transfer function of the electrical circuit shown below.
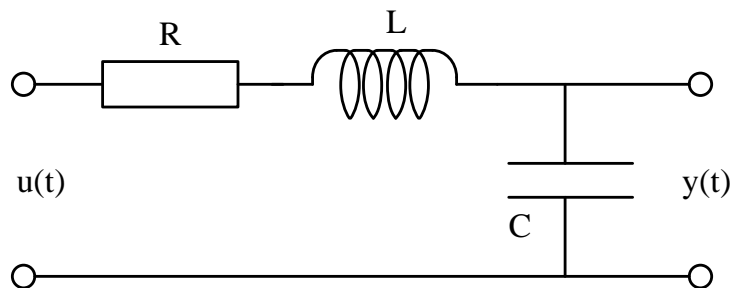

For R = 1 Ω, L = 0.5 H, C = 0.5 F, and a unit step input u(t) with zero initial conditions, compute y(t). Sketch the time function y(t) and plot it with MATLAB.

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4. In the mechanical system shown in the figure below, m is the mass, k is the spring constant, b is the friction constant, u(t) is the external applied force and y(t) is the corresponding displacement. Find the transfer function of this system.

For m = 1 kg, k = 1 kg/s^2, b = 0.5 kg/s, and a step input u(t) = 2 N, compute the response y(t) and plot it with MATLAB.
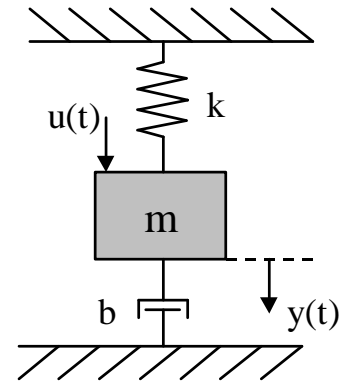
5. Write down the transfer function Y(s)/R(s) of the following block diagram.
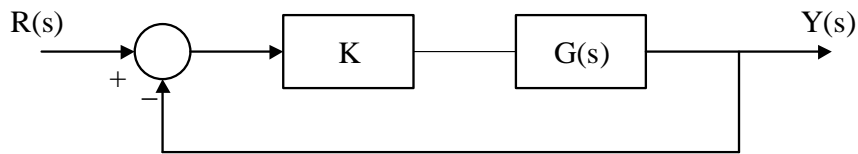
a) For G(s) = 1/(s + 10) and K = 10, determine the closed loop transfer function with MATLAB.
b) For K = 1, 5, 10, and 100, plot y(t) on the same window for a unit-step input r(t) with MATLAB, respectively. Comment on the results.
c) Repeat (b) with a unit-impulse input r(t).

6. Find the closed loop transfer function for the following diagram.
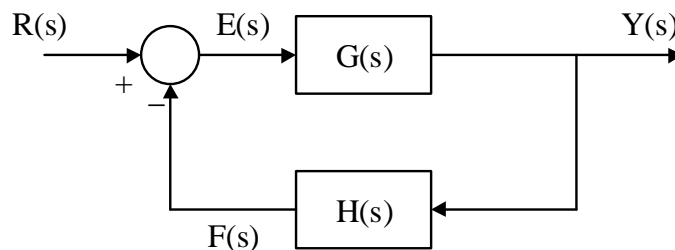
a) For G(s) = 8/(s^2 + 7s + 10) and H(s) = s+2, determine the closed loop transfer function with MATLAB.
b) Plot y(t) for a unit-step input r(t) with MATLAB.

7. Determine the transfer function of the following diagram. Check your answer with MATLAB.

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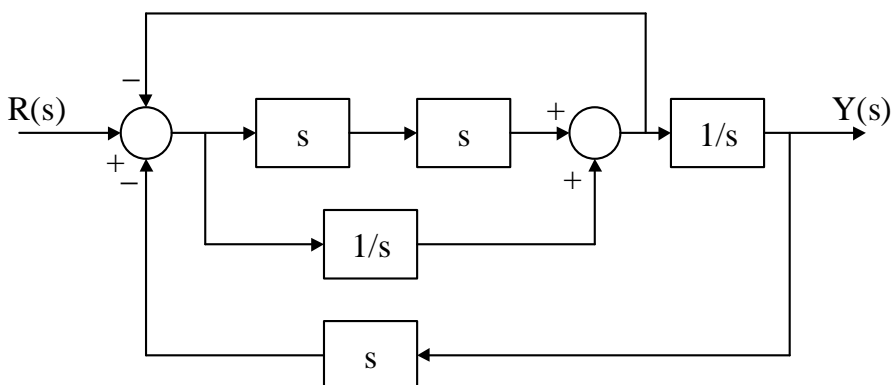
8. Determine the transfer function of the following diagram.
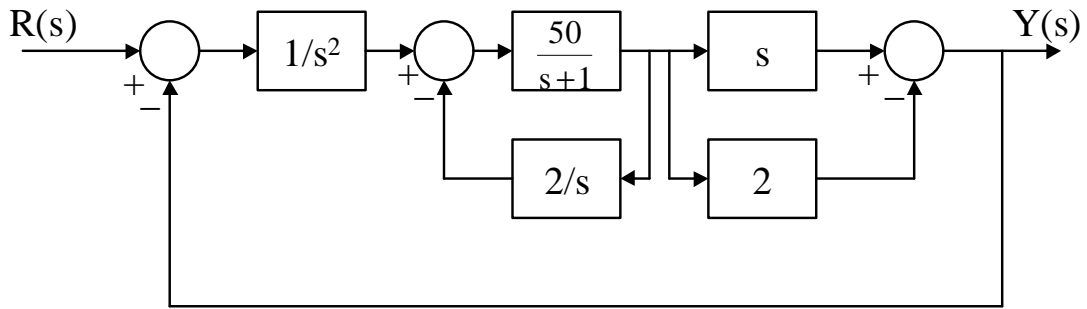
a) Check you result with MATLAB.
b) Plot y(t) for a unit-impulse input r(t) with MATLAB.

9. Determine the total output Y(s) for the following system.